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Dispersion Relation for the Regge Parameters

HUNG CHENG
Physics Department, California Institute of Technology, Pasadena, California
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The Regge parameters $\alpha(s)$ and $\beta(s)$ of trajectories which cross one another at some point s_0 are proved to form different branches of an analytic function with the branch point at s_0. A new set of Regge parameters are found which are regular in the neighborhood of s_0. Dispersion relations written for these parameters thus take a simple form and should be the ones to use for the dynamical determination of the Regge parameters.

In the example of nonrelativistic scattering by a superposition of Yukawa potentials, crossing of trajectories is shown to happen for an infinite number of Regge trajectories. We also give a method to obtain the asymptotic series of $\alpha(s)$ and $\beta(s)$ for large s, in the nonrelativistic case, and from this we show that in the potential

$$\int_{m^2}^{\infty} \sigma(\mu^2)\frac{e^{-\mu r}}{r}d\mu^2,$$

all $\alpha(s)$ and $\beta(s)e^{-i\pi\alpha(s)}$ are real analytic functions of s.

I. CROSSING OF TRAJECTORIES

THERE has been the hope[1] that dispersion relations satisfied by the Regge parameters $\alpha(s)$ and $\beta(s)$ together with the unitarity conditions, may help to determine these parameters in a dynamical way. This approach faces, however, the difficulty that when Regge trajectories accidentally cross, extra branch cuts for $\alpha(s)$ and $\beta(s)$ may arise. Since there is no way to determine where the crossings occur without actually knowing all $\alpha(s)$ themselves, the feasibility of this method was controversial.

It is the purpose of this paper to point out that a new set of Regge parameters, all of them algebraic functions of $\alpha(s)$ and $\beta(s)$, can be introduced which do not have branch cuts arising from crossing of trajectories. Dispersion relations satisfied by these parameters should therefore be used for dynamical calculation, instead of those satisfied by $\alpha(s)$ and $\beta(s)$, which can be used only when the effect of crossing is neglected.

We first note that the $\alpha(s)$ are the roots of the analytic function $1/A(l,s)$, where $A(l,s)$ is the partial wave amplitude with s the c.m. energy squared in the relativistic case, and the laboratory energy in the nonrelativistic case. Writing $D(l,s)=1/A(l,s)$, we find

$$D(\alpha(s),s)=0. \quad (1)$$

If $D(l,s)$ is regular in the neighborhood of (α_0,s_0), and $D(\alpha_0,s_0)=0$ then we can expand (1) in a Taylor series in this neighborhood, obtaining

$$a[\alpha(s)-\alpha_0]+b(s-s_0)+c[\alpha(s)-\alpha_0]^2 +d(s-s_0)^2+e(s-s_0)[\alpha(s)-\alpha_0]+\cdots=0, \quad (2)$$

where $a=\partial D(l,s)/\partial l$, $b=\partial D(l,s)/\partial s$, evaluated at (α_0,s_0). If $a\neq 0$, we have from (2)

$$\alpha(s)=\alpha_0-(b/a)(s-s_0)+\cdots, \quad (3)$$

and $\alpha(s)$ is analytic in the neighborhood of s_0. However, if $a=0$ and $c\neq 0$, then we have from (2)

$$\alpha(s)=\alpha_0\pm(-b/c)^{1/2}(s-s_0)^{1/2}+\cdots; \quad (4)$$

thus, there are two Regge trajectories which cross at s_0 and both of them have s_0 as a branch point. It should be noticed that if $b=0$, $\alpha(s)$ need not have a branch point at s_0. Crossing of trajectories does not always give rise to a branch point for the Regge parameters $\alpha(s)$ and $\beta(s)$.

Consider now two trajectories $\alpha_1(s)$ and $\alpha_2(s)$ crossing at s_0, which is a branch point for the functions $\alpha_1(s)$ and $\alpha_2(s)$. Then we can write

$$\alpha_1(s)=(s-s_0)^{1/2}f_1(s)+f_2(s),$$
$$\alpha_2(s)=-(s-s_0)^{1/2}f_1(s)+f_2(s), \quad (5)$$

where, $f_1(s)$ and $f_2(s)$ are both regular in the neighborhood of s_0. Thus, both of the functions $\alpha_1(s)+\alpha_2(s)$ and $\alpha_1(s)\alpha_2(s)$ are regular in the neighborhood of s_0. Dispersion relations written for these two functions do not have to take account of the branch cut arising from crossing of Regge trajectories. One may also note that in the complex plane u, $u=(s-s_0)^{1/2}$, $\alpha_1(u)$ and $\alpha_2(u)$ are both regular in the neighborhood of the origin, and satisfy the relation

$$\alpha_1(u)=\alpha_2(-u). \quad (6)$$

The fermion Regge trajectories can also be discussed from this view point.[2]

Assuming that $\alpha_1(s)$ and $\alpha_2(s)$ are real analytic functions of s, which we shall shortly see to be true for all trajectories in nonrelativistic scattering by a superposition of Yukawa potentials, then if $\alpha_1(s)$ and $\alpha_2(s)$ cross at s_0 they also cross at s_0^*. In analogy to (5) we can again write in the neighborhood of s_0^*,

$$\alpha_1(s)=(s-s_0^*)^{1/2}f_3(s)+f_4(s),$$
$$\alpha_2(s)=-(s-s_0^*)^{1/2}f_3(s)+f_4(s), \quad (7)$$

where $f_3(s)$ and $f_4(s)$ are both regular in the neighbor-

[1] H. Cheng and D. Sharp (to be published).

[2] V. N. Gribov (to be published).

hood of s_0^*. From (7) we get

$$f_4(s)=\tfrac{1}{2}[\alpha_1(s)+\alpha_2(s)]=f_2(s),$$
$$f_3(s)=\frac{\alpha_1(s)-\alpha_2(s)}{2(s-s_0^*)^{1/2}}=\left(\frac{s-s_0}{s-s_0^*}\right)^{1/2}f_1(s). \tag{8}$$

If $s_0\neq s_0^*$, combining (6), (7), and (8) gives

$$\begin{aligned}\alpha_1(s)&=[(s-s_0)(s-s_0^*]^{1/2}g(s)+f_2(s),\\ \alpha_2(s)&=-[(s-s_0)(s-s_0^*)]^{1/2}g(s)+f_2(s),\end{aligned} \tag{9}$$

where $g(s)$ and $f_2(s)$ are both regular in the neighborhoods of s_0 and s_0^*. The crossing of $\alpha_1(s)$ and $\alpha_2(s)$ at s_0 and s_0^* thus produces a branch cut from s_0 to s_0^*, if s_0 is complex, and from s_0 to ∞ if s_0 is real, for both $\alpha_1(s)$ and $\alpha_2(s)$. The function $[\alpha_1(s)-\alpha_2(s)]/[(s-s_0)(s-s_0^*)]^{1/2}$ is also regular in the neighborhood of both s_0 and s_0^*. If $\alpha_1(s)$ and $\alpha_2(s)$ do not cross with other trajectories and if s_0 is known, dispersion relations written for this function also take a simple form.

Let us write

$$A(l,s)=N_1(l,s)/[l-\alpha_1(s)][l-\alpha_2(s)], \tag{10}$$

then $N_1(l,s)$ is regular in the neighborhood of (α_0,s_0) and (α_0^*,s_0^*). We have, as before,

$$\begin{aligned}b_1(s)&=N_1(\alpha_1(s),s)\\ &=[(s-s_0)(s-s_0^*)]^{1/2}h_1(s)+h_2(s), \qquad (11)\\ b_2(s)&=N_1(\alpha_2(s),s)\\ &=-[(s-s_0)(s-s_0^*)]^{1/2}h_1(s)+h_2(s),\end{aligned}$$

with $h_1(s)$ and $h_2(s)$ both regular in the neighborhood of s_0 and s_0^*. The functions $b_1(s)+b_2(s)$ and $b_1(s)b_2(s)$ are thus regular in the neighborhoods of s_0 and s_0^*. The residue of $A(l,s)$ at $\alpha_1(s)$ which we shall call $r_1(s)$, can be obtained from (9), (10), and (11) as

$$r_1(s)=\frac{1}{2[(s-s_0)(s-s_0^*)]^{1/2}}\frac{h_2(s)}{g(s)}+\frac{1}{2}\frac{h_1(s)}{g(s)}, \tag{12}$$

and, similarly,

$$r_2(s)=-\frac{1}{2[(s-s_0)(s-s_0^*)]^{1/2}}\frac{h_2(s)}{g(s)}+\frac{h_1(s)}{g(s)}. \tag{13}$$

A similar expression holds for $\beta(s)$, which is equal to $-\pi[2\alpha(s)+1]r(s)$. We note from (12) and (13) that both $r_1(s)$ and $r_2(s)$ are infinite at s_0 and s_0^*, and for this reason it is more convenient to discuss the quantities $b_1(s)$ and $b_2(s)$. It should be remembered that in (9), (11), (12), and (13), the factor $[(s-s_0)(s-s_0^*)]^{1/2}$ is replaced by $(s-s_0)^{1/2}$ if s_0 is real.

We also see from (9) that on the two sides of the branch cut from s_0 to s_0^*

$$\begin{aligned}\alpha_1(s+\epsilon)&=\alpha_2(s-\epsilon),\\ \alpha_2(s+\epsilon)&=\alpha_1(s-\epsilon).\end{aligned} \tag{14}$$

This result can also be obtained by arguing that since $A(l,s)$ does not have a branch cut there, the two sets of Regge poles on the two sides of this branch cut are the same. Similarly

$$\begin{aligned}b_1(s+\epsilon)&=b_2(s-\epsilon),\\ b_2(s+\epsilon)&=b_1(s-\epsilon).\end{aligned} \tag{15}$$

If three Regge trajectories $\alpha_1(s)$, $\alpha_2(s)$, and $\alpha_3(s)$ cross at s_0, several possibilities arise. (i) All three functions are regular at s_0; (ii) One of the functions is regular at s_0, and the other two have s_0 as a branch point (this case is the same as the one already treated). (iii) All three functions have s_0 as a branch point. In the last case, we shall have

$$\begin{aligned}\alpha_1(s+\epsilon)&=\alpha_2(s-\epsilon),\\ \alpha_2(s+\epsilon)&=\alpha_3(s-\epsilon),\\ \alpha_3(s+\epsilon)&=\alpha_1(s-\epsilon),\end{aligned} \tag{16}$$

when s is at the branch cut. Thus, $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$ form three branches of an analytic function with the branch point at s_0. In general, if n Regge trajectories cross at s_0 and s_0^*, and form n branches of an analytic function, it can be shown that we may write

$$\begin{aligned}\alpha_1(s)=&F_1(s)+(s-s_0)^{1/n}(s-s_0^*)^{(n-1)/n}F_2(s)+\cdots\\ &+(s-s_0)^{(n-1)/n}(s-s_0^*)^{1/n}F_n(s),\\ \alpha_2(s)=&F_1(s)+e^{2\pi i/n}(s-s_0)^{1/n}(s-s_0^*)^{(n-1)/n}F_2(s)+\cdots\\ &+e^{2\pi i(n-1)/n}(s-s_0)^{(n-1)/n}(s-s_0^*)^{1/n}F_n(s),\\ &\cdots\\ \alpha_n(s)=&F_1(s)+e^{2\pi i/n(n-1)}(s-s_0)^{1/n}\\ &\times(s-s_0^*)^{(n-1)/n}F_2(s)+\cdots\\ &+e^{2\pi i[(n-1)/n](n-1)}(s-s_0)^{(n-1)/n}(s-s_0^*)^{1/n}F_n(s).\end{aligned} \tag{17}$$

In (17) the functions $F_1(s)$, $F_2(s)$, $\cdots F_n(s)$ are regular in the neighborhoods of s_0 and s_0^*. We see from (17) that a branch out from s_0 and s_0^* exists for each of the functions $\alpha_1(s)$, $\alpha_2(s)$, $\cdots\alpha_n(s)$.

The functions $F_1(s)$, $F_2(s)$, $\cdots F_n(s)$ can be easily expressed in terms of the α functions. We have

$$\begin{aligned}F_l(s)=&(s-s_0)^{-(l-1)/n}(s-s_0^*)^{-(n-l+1)/n}\\ &\times\frac{1}{n}\sum_{j=1}^{n}e^{-[2\pi(l-1)/n](j-1)i}\alpha_j(s),\quad l\neq 1,\\ F_1(l)=&\frac{1}{n}\sum_{j=1}^{n}\alpha_j(s).\end{aligned} \tag{18}$$

Similar expressions for the β function can be obtained in the same way and will be omitted.

The prescription for finding a set of Regge parameters which are real analytic functions of s without irregular branch cuts can now be stated. Suppose there are altogether m Regge poles in the amplitude $A(l,t)$, and for simplicity of argument we assume m finite (in practice we are only able to take care of a finite number of

Regge poles anyway). Let us define

$$a_1(s)=\sum_{i=1}^{m}\alpha_i(s),$$

$$a_2(s)=\sum_{i=1}^{m}\sum_{j<i}\alpha_i(s)\alpha_j(s),$$

$$a_3(s)=\sum_{i=1}^{m}\sum_{j<i}\sum_{k<j}\alpha_i(s)\alpha_j(s)\alpha_k(s),$$

$$a_m(s)=\prod_{i=1}^{m}\alpha_i(s).$$

The parameters $a_1(s)$, $a_2(s)\cdots a_m(s)$ then have no branch cuts arising from the crossing of Regge trajectories. To prove this, we write

$$A(l,s)=N(l,s)/\prod_{i=1}^{m}[l-\alpha_i(s)],\tag{20}$$

where $N(l,s)$ has no pole. When crossing of two or more Regge trajectories occurs, branch cuts may arise for two or more Regge trajectories. But $A(l,s)$ does not possess this branch cut, hence the two sets of Regge poles at the two sides of the branch cut have to be the same. In other words, one set must be a permutation of the other. From (19) we see that $a_1(s)$, $a_2(s)$, $\cdots a_m(s)$ are invariant upon permutation. Thus, they do not have branch cuts arising from crossing of trajectories.

In the same manner, let us first define

$$\begin{aligned}b_1(s)&=N(\alpha_1(s),s)e^{-i\pi\alpha_1(s)},\\ b_2(s)&=N(\alpha_2(s),s)e^{-i\pi\alpha_2(s)},\\ &\cdots\\ b_m(s)&=N(\alpha_m(s),s)e^{-i\pi\alpha_m(s)}.\end{aligned}\tag{21}$$

The factor $e^{-i\pi\alpha(s)}$ is added to make all b functions real analytic when $\alpha(s)$ is real analytic. Then let

$$\begin{aligned}c_1(s)&=\sum_{i=1}^{m}b_i(s),\\ c_2(s)&=\sum_{i=1}^{m}\sum_{j<i}b_i(s)b_j(s),\\ &\cdots\\ c_m(s)&=\prod_{i=1}^{m}b_i(s).\end{aligned}\tag{22}$$

By the same argument the $c_i(s)$ functions can be shown to possess no branch cut arising from the crossing of trajectories. In the case of scattering by a superposition of Yukawa potential, for instance, the dispersion relation

$$f(s)=\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}f(s')}{s'-s}ds'+f(\infty),$$

is satisfied by all of the $a_i(s)$ and $c_i(s)$ functions.

To express $\alpha_i(s)$ in terms of $a_i(s)$, we first note that

$$\prod_{i=1}^{m}[l-\alpha_i(s)]=l^m-a_1(s)l^{m-1}+a_2(s)l^{m-2}+\cdots+a_m(s).\tag{23}$$

Equating the right side of (23) to zero and solving the mth-order polynomial will give m roots, which are precisely $\alpha_1(s)$, $\alpha_2(s)$, $\cdots\alpha_m(s)$. Since the coefficients of this polynomial are real analytic functions of s, it is once again obvious that the $\alpha_i(s)$ are real analytic functions of s, and when some of the roots coincide, a branch cut of the form $(s-s_0)^{1/n}$ would arise. The same holds, needless to say, for $b_i(s)$ and $c_i(s)$.

II. REGGE TRAJECTORIES IN THE SCHRÖDINGER EQUATION

In this section we turn to the Schrödinger equation in the potential

$$\int_{m^2}^\infty\sigma(\mu^2)\frac{e^{-\mu r}}{r}d\mu^2,$$

which takes the form

$$[d^2/dr^2+k^2-l(l+1)/r^2]\psi(k,l,r)=V(r)\psi(k,l,r).\tag{24}$$

For large r, the right side of (24) is exponentially small, and for small r, it is small compared to the term $[l(l+1)/r^2]\psi(k,l,r)$. As k^2 approaches infinity, therefore, the right side will be small for all r and can be treated as a perturbation. We, thus, write

$$\psi(k,l,x)=x^{1/2}J_\lambda(kx)+\frac{\pi i}{4}\int_0^x(xy)^{1/2}\times[H_\lambda^{(2)}(kx)H_\lambda^{(1)}(ky)-H_\lambda^{(2)}(ky)H_\lambda^{(1)}(kx)]\times V(y)\psi(k,l,y)dy,\tag{25}$$

where $\lambda=l+\frac{1}{2}$. The right side of (25) is defined by analytic continuation if the integral does not converge at the lower limit. From (25), we easily obtain

$$f(l,s)=\frac{1}{2k}\left[-\pi\int_0^\infty y^{1/2}J_\lambda(ky)V(y)\psi(k,l,y)dy\right]\Big/\left[1+\tfrac{1}{2}\pi i\int_0^\infty y^{1/2}H_\lambda^{(1)}(ky)V(y)\psi(k,l,y)dy\right].\tag{26}$$

Substituting $\psi(k,l,y)=y^{1/2}J_\lambda(ky)$ in (26), we obtain the first Born approximation

$$f_B(l,s)=\left\{-\int_{m^2}^\infty\frac{d\mu^2}{2s}\sigma(\mu^2)Q_l\left(1+\frac{\mu^2}{2s}\right)\right\}\Big/\left\{1+\frac{1}{2\cos l\pi}\int_{m^2}^\infty d\mu^2\,\sigma(\mu^2)\left[i\frac{e^{-il\pi}}{k}Q_l\left(1+\frac{\mu^2}{2s}\right)+\int_0^{\pi/2}\frac{\cos(2l+1)\varphi d\varphi}{(\frac{1}{4}\mu^2+s\cos^2\varphi)^{1/2}}\right]\right\}.\tag{27}$$

The Regge pole trajectories are determined by the vanishing of the denominator in (27). Thus,

$$D(\alpha,s)=1+\frac{1}{2\cos\alpha\pi}\int_{m^2}^{\infty}d\mu^2\,\sigma(\mu^2)\left[i\frac{e^{-i\alpha\pi}}{k}Q_\alpha\left(1+\frac{\mu^2}{2s}\right)+\int_0^{\pi/2}\frac{\cos(2\alpha+1)\varphi d\varphi}{(\frac{1}{4}\mu^2+s\cos^2\varphi)^{1/2}}\right]=0. \quad (28)$$

For very large k, the second term in (28) is small compared to 1 unless α is near $-1,\ -2,\ \cdots,\ -n,\ \cdots$ where Q_α has a pole. [At $\alpha=(n+\frac{1}{2})$, although $\cos\alpha\pi=0$,

$$\frac{ie^{-i\alpha\pi}}{k}Q_\alpha\left(1+\frac{\mu^2}{2s}\right)+\int_0^{\pi/2}\frac{\cos(2\alpha+1)\varphi d\varphi}{(\frac{1}{4}\mu^2+s\cos^2\varphi)^{1/2}}$$

also turns out to vanish.] Thus, the Regge poles approach $-1,\ -2,\ \cdots-n$ st large s. After some algebra we find

$$\alpha_n(s)=-n-1-\frac{g^2}{2\eta}+\frac{g^4}{4\eta^2}\ln\eta+\cdots,$$
$$\beta_n(s)=\frac{g^2\pi}{2\eta^2}\left[2n+\tfrac{1}{2}+\frac{g^2}{\eta}-(2n+1)\frac{g^2}{2\eta}\ln\eta+\cdots\right], \quad (29)$$

where

$$g^2=\int_{m^2}^{\infty}6(\mu^2)d\mu^2 \quad\text{and}\quad \eta=-ik.$$

The first two terms of $\alpha_n(s)$ and the first term of $\beta_n(s)$ agree with the behavior of $\alpha_n(s)$ and $\beta_n(s)$ in a Coulomb potential, as was concluded earlier.[3] Higher order terms for $\alpha(s)$ and $\beta(s)$ at high energy can be obtained by taking into account the second and even higher order Born approximation.

Several interesting properties of Regge poles can now be obtained. First, since there is an infinite number of Regge poles approaching the point $l=-\frac{1}{2}$ in complex-conjugate pairs, at negative energy,[4] each pair would remain complex-conjugate to each other at all negative energies, until they cross. Therefore, if this pair of trajectories does not cross, they would both approach the point $-n$ at high, negative energy in complex-conjugate pairs. However, it can be seen that

$$\partial D(\alpha,s)/\partial\alpha\neq 0$$

at large s, hence there is only one pole approaching each $-n$. Thus each member of the complex-conjugate pairs of trajectories cross its complex-conjugate partner at some negative energy. The Regge poles in the left-hand plane thus behave in the following way: each of them comes out from a point $-n$ at infinite energy, and stays on the real axis, as can be seen from (29), until it crosses with another trajectory. Then both of them may leave the real axis and become complex-conjugate pairs at negative energy. All $\alpha(s)$ and $\beta(s)e^{-i\pi\alpha(s)}$ are, therefore, real analytic functions of s.

Finally, we mention that in the nonrelativistic case, the Regge trajectories do not cross in the right-hand plane $\mathrm{Re}\,l\geq -\frac{1}{2}$ when the energy is negative. To see this we observe that if two Regge poles cross at (α_0,s_0), $s_0<0$, then both ψ and $\partial\psi/\partial l$ vanish exponentially for larger r at this point. Differentiating (24) with respect to l, multiplying it by ψ and integrating from 0 to ∞ will give us the desired result.

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[3] H. Cheng, Phys. Rev. **127**, 647 (1962).

[4] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962); Kenneth Wilson (private communication to M. Gell-Mann).